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Note

On modified Chebyshev polynomials

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Abstract

Two elegant representations are derived for the modified Chebyshev polynomials discussed by Witula and Slota [R. Witula, D. Slota, On modified Chebyshev polynomials, J. Math. Anal. Appl. 324 (2006) 321–343].

Keywords: Modified Chebyshev polynomials; Student's t random variable

1. Introduction

The nth modified Chebyshev polynomials of the first kind and the second kind are defined by

$$\Omega_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} x^{n-2k} \tag{1}$$

and

$$V_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} x^{n-2k}, \tag{2}$$

respectively. The recent paper by Witula and Slota [1] provided a detailed analysis of the properties of (1) and (2). Here, we would like to derive two more properties of (1) and (2) that appear to have been overlooked by [1].

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2. Main results

Let Z be a random variable that has the Student's t distribution with $2n$ degrees of freedom, see Johnson et al. [2,3] for details. Note that the $2k$th moment of Z can be written as

$$\begin{aligned}
E\left[Z^{2k}\right] &= (2n)^k \frac{\Gamma((2n-2k)/2)\Gamma((2k+1)/2)}{\Gamma(1/2)\Gamma(n)} \\
&= (2n)^k \frac{\Gamma(n-k)\Gamma(k+1/2)}{\sqrt{\pi}\,\Gamma(n)} \\
&= n^k \frac{\Gamma(n-k)k!(1\cdot 3\cdot 5\cdots(2k-1))}{k!(n-1)!} \\
&= n^k 2^{-k} \frac{\Gamma(n-k)(2k)!}{k!(n-1)!} \\
&= n^k 2^{-k} \frac{(n-k-1)!(2k)!}{k!(n-1)!} \\
&= n^k 2^{-k} \frac{n}{n-k} \binom{n-k}{k} \frac{(n-2k)!(2k)!}{n!}
\end{aligned}$$

for $k \geqslant 1$. Thus, noting that the odd moments of Z are zero, one can rewrite (1) as

$$\begin{aligned}
\Omega_n(x) &= \sum_{k=0}^{[n/2]} (\sqrt{2}i/\sqrt{n})^{2k} E\left[Z^{2k}\right] \binom{n}{2k} x^{n-2k} \\
&= E\left[\sum_{k=0}^{n} (\sqrt{2}i/\sqrt{n})^{k} Z^{k} \binom{n}{k} x^{n-k}\right] \\
&= E\left[x + \frac{\sqrt{2}iZ}{\sqrt{n}}\right]^n,
\end{aligned}$$

an elegant representation as the expectation of the power of a Student's t random variable. Similarly, one can rewrite (2) as

$$V_n(x) = E\left[x + \frac{\sqrt{2}iZ}{\sqrt{n+1}}\right]^n,$$

where Z is now a Student's t random variable with $2n+2$ degrees of freedom. As usual, i denotes the complex unit $i = \sqrt{-1}$.

Note that one can derive similar probabilistic representations for other polynomials. For example, Bernstein polynomials can be expressed as expectations using binomial random variables, see [4–6].

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